RELATIVITY AND COSMOLOGY I

Problem Set 10 — Fall 2023

1. The Precession of Mercury

Consider the motion of Mercury in the Solar System. Ignoring the negligible backreaction by all the planets, the metric around the Sun is given by the Schwarzschild solution:

$$-\mathrm{d}\tau^2 = -\left(1-\frac{r_s}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{r_s}{r}}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{1}$$

with $r_s = 2MG$, and M is the mass of the Sun.
The goal of this exercise is to find how the motion of Mercury deviates from the one predicted by Newtonian gravity and by Kepler's laws.

(a) In Newtonian mechanics, show that conservation of energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ for a planet of mass m in orbit around a star of mass M implies

$$\left(\frac{dr}{dt}\right)^2 = \frac{2\mathcal{E}}{m} + \frac{r_s}{r} - \frac{\mathcal{L}^2}{m^2r^2} \tag{2}$$

Express this as a second order differential equation in $u(\phi)$ where $u = r^{-1}$.

(b) Analogously to what you did in Problem Set 9, use the Killing vectors of (1) to show that timelike geodesics in the Schwarzschild metric satisfy a similar equation to (2). By direct comparison, how are the conserved quantities $E \equiv g_{tt}\frac{dt}{d\tau}$ and $L \equiv g_{\phi\phi}\frac{d\phi}{d\tau}$ related to $\mathcal{E}$ and $\mathcal{L}$? Do the same change of variables as in (a) and show that

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} + \frac{3r_s}{2}u^2\,. \tag{3}$$

What is the difference between (3) and what you found at the end of (a)?

(c) Show that $u(\phi) = u(0)(1 + e\cos\phi)$ is a solution in the Newtonian case. For which values of the constant e is this a bound orbit? How are the constants $u(0)$ and e related to the perihelion distance r_{min} and the aphelion distance r_{max}?

Let us now estimate the precession of the orbit due to the quadratic term in (3). To start, let us assume that the orbit is approximately circular, $(u-\bar{u})^2 \ll \bar{u}^2$ with $\bar{u}$ being the average of u over the orbit.

(d) Linearize (3) and show that

$$u(\phi) = u_0(1 + e\cos(K\phi)) \tag{4}$$

is a solution to the linearized (3). Find the coefficient K.

(e) Show that any relativistic orbit thus precesses by

$$\Delta\phi \approx 6\pi GM\bar{u}\,, \tag{5}$$

where we have neglected terms of higher order in $GM\bar{u}$.[1]

(f) Given that the mass of the Sun is $M = 1.99 \times 10^{30}$ kg, and that the 88-day orbit of Mercury has $r_{\text{min}} = 4.60 \times 10^{10}$ m and $r_{\text{max}} = 6.98 \times 10^{10}$ m, show that the orbit of Mercury precesses by 43 arcseconds per century.

2. Light Deflection in General Relativity

In this exercise we will study the deflection of a light-ray coming from far away due to the presence of a massive body close to the trajectory, such as a star, galaxy or black hole.

(a) First, use the symmetries of the Schwarzschild geometry to find the equations of motion of light-like geodesics

$$E^2 = \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{r_s}{r}\right)\frac{L^2}{r^2}, \tag{6}$$

where $r_s = 2GM$ is the Schwartzschild radius.

(b) To study the deflection of light it is convenient to consider the coordinate r as a function of the angle ϕ. Using you answer at point a), find the equations of motion for $r(\phi)$. Integrate them to find $\phi(r)$ (do not try to compute the integral yet).

(c) We want to trade the conserved quantities E and L for quantities more suited to the problem. Find a relation between the distance of closest approach r_0, E and L. Then express the deflection angle $\Delta\phi$ in terms of $\phi(r_0)$.

(d) The integral computing $\phi(r_0)$ is messy, but we can study it perturbatively by assuming $r_s/r_0 \ll 1$. Show that the leading order term yields the result of Newtonian mechanics, i.e. no light deflection.

(e) Study the subleading order to show that the first correction to the Newtonian result is[2]

$$\Delta\phi = \frac{4GM}{r_0} \tag{7}$$

[1] Why is that a good approximation?

[2] Actually you could argue that in the limit $m \to 0$ the Newtonian equation of motion $m\vec{a} = -\frac{GmM}{r^3}\vec{r}$ is well behaved and that photons still interact gravitationnaly. You can show that the prediciton for the deflection angle in this case is $\Delta\phi = 2GM/r_0$, half of the GR result.